

# DISCRETE UNCERTAINTY PRINCIPLES AND VIRIAL IDENTITIES

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**ABSTRACT.** In this paper we review the Heisenberg uncertainty principle in a discrete setting and, as in the classical uncertainty principle, we give it a dynamical sense related to the discrete Schrödinger equation. We study the convergence of the relation to the classical uncertainty principle, and, as a counterpart, we also obtain another discrete uncertainty relation that does not have an analogous form in the continuous case. Moreover, in the case of the Discrete Fourier Transform, we give a inequality that allows us to relate the minimizer to the Gaussian.

## 1. INTRODUCTION

The well-known Heisenberg uncertainty principle [3] states that

$$(1) \quad \frac{2}{d} \left( \int_{\mathbb{R}^d} |xf(x)|^2 dx \right)^{1/2} \left( \int_{\mathbb{R}^d} |\nabla f(x)|^2 dx \right)^{1/2} \geq \int_{\mathbb{R}^d} |f(x)|^2 dx.$$

Moreover, the minimizing function (that for which (1) is an equality) verifies, for  $\alpha > 0$ ,  $\nabla f(x) + \alpha x f(x) = 0 \implies f(x) = Ce^{-\alpha|x|^2/2}$  (Gaussian).

Now, if we consider  $u(x, t)$  a solution to the Schrödinger free equation, there is a dynamic interpretation of the uncertainty principle, which was exploited in [4, 5].

**Theorem 1.1** (Dynamic uncertainty principle). *Assume  $u(x, t)$  is a solution to*

$$\begin{cases} \partial_t u(x, t) = i\Delta u(x, t), & x \in \mathbb{R}^d, t \in \mathbb{R}, \\ u(x, 0) = u_0(x), \end{cases}$$

where  $u_0 \in \dot{H}^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d, |x|^2 dx)$ ,  $\|u_0\|_2^2 = 2/d$ . For a real function  $\phi(x)$  we define

$$h(t) = \int_{\mathbb{R}^d} \phi(x) |u(x, t)|^2 dx, \quad a = \int_{\mathbb{R}^d} |x|^2 |u_0(x)|^2 dx < +\infty, \quad b = \int_{\mathbb{R}^d} |\nabla u_0(x)|^2 dx < +\infty.$$

Then,

$$(2) \quad \ddot{h}(t) = 4 \int_{\mathbb{R}^d} \nabla u D^2 \phi \overline{\nabla u} - \int_{\mathbb{R}^d} \Delta^2 \phi |u|^2. \quad (\text{Virial identity})$$

Moreover, if  $\phi(x) = |x|^2$ ,

$$(3) \quad h(t) = a + 4bt^2 \geq a + \frac{4t^2}{a},$$

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and, if these two convex parabolas intersect each other, they are the same parabola and the initial datum is  $u_0(x) = Ce^{-\alpha|x|^2/2}$ , being then

$$u(x, t) = \left( \frac{1}{2\alpha it + 1} \right)^{d/2} e^{\frac{i\alpha^2 t |x|^2}{4\alpha^2 t^2 + 1} - \frac{\alpha |x|^2}{8\alpha^2 t^2 + 2}}.$$

Observe that the normalization condition in the initial datum gives, thanks to the uncertainty principle (1) that  $ab \geq 1$ .

In this paper we want to develop this theory in a discrete setting discretizing the momentum and position operators. Since we can relate a sequence to a periodic function via Fourier series, there is a duality between discrete uncertainty principles and periodic uncertainty principles. The relation we study here appears in the literature (see [1, 7, 2]) in this periodic form. Moreover, in [2] the authors suggested another uncertainty relation. Their aim was to study the angular momentum - angle variables on the sphere, so they related the orbital angular momentum to the azimuthal angle about the  $z$  axis. Then, the orbital momentum is written as a differential operator and, for a meaningful uncertainty principle, periodicity is required for the position operator. Hence, the authors suggested the operators  $\cos(x)$  and  $\sin(x)$  to represent position. Considering this duality via Fourier series, the second case is connected with the discrete version of Heisenberg uncertainty principle that we will study here. In the first case, we will get another relation that does not have a continuous version.

Another version of the Heisenberg uncertainty principle appears in [10, 11, 7], but in this case the equality is not attained. However, it is possible to construct a sequence of polynomials  $p_k$  of degree  $k$  such that the inequality approaches the equality as  $k$  tends to infinity. Nevertheless, we will not study this relation here.

As it happens for the Heisenberg uncertainty principle in the continuous case, we will derive Virial identities equivalent to (2) for both relations. Thus we will give them a dynamical interpretation (equivalent to (3)). On the one hand, the dynamics will be given by the discrete Schrödinger equation, as it is expected. On the other hand, it will appear an equation that turns out to be an  $L^2$ -invariant factorization of the one dimensional wave equation.

Since we see an analogy between the continuous and discrete dynamic uncertainty principles, it seems reasonable to have similarities between the solution to the continuous Schrödinger equation with initial datum the Gaussian and the solution to the discrete equation, now with initial datum the minimizer of the discrete relation. In the continuous case, it is known that this solution verifies another equation in the form  $(\mathcal{S} + \mathcal{A})\omega = 0$ , where  $\mathcal{S}$  is a symmetric operator and  $\mathcal{A}$  is a skew-symmetric operator, so we prove here that in the discrete case the same statement holds.

Apart from this, we consider another discrete setting, the case of finite sequences. The motivation here comes from [8], where the author gives a relation for the Discrete Fourier Transform, but he suggested that the minimizing sequence of his inequality is not similar to the Gaussian. Here, we will slightly modify this relation in order to see that the minimizer approaches the minimizer of the periodic uncertainty we have mentioned above. Besides, we give two uncertainty principles truncating the operators we will study in Section 2 and imposing periodic or Dirichlet boundary conditions. In these two cases, when the number of nodes tends to infinity we recover the discrete uncertainty principle. However, we will see that we do not have a convex parabola with these versions of the position and momentum operators. This fact is consistent with the

periodic Schrödinger equation, since there is no convex parabola equivalent to  $h(t)$  in Theorem 1.1 in this case.

This paper is organized as follows: In Section 2 we introduce the discrete uncertainty principle we want to study, seeing that the minimizer tends to the Gaussian in the continuous limit. We also discuss the other discrete uncertainty principle related to the  $\cos(x)$  operator in the space of periodic functions. In Section 3 we give dynamical interpretations for the uncertainty principles discussed in Section 2. In Section 4 we observe that the continuous and discrete solutions to the Schrödinger equation with initial datum the respective minimizer share some properties. In Section 5 first we give a slightly modification of the uncertainty principle stated in [8] that allows us to connect the minimizer to the minimizer of the periodic uncertainty principle of Section 2, and therefore, to the Gaussian. We also truncate the position and momentum operators in Section 2 to consider two cases, the periodic and the Dirichlet case, noticing that we can not repeat the theory we develop in Section 3.

## 2. UNCERTAINTY PRINCIPLE IN $H_h(\mathbb{Z}^d)$

A useful tool to obtain uncertainty relations is the following (see [6]): Let  $\mathcal{S}$  a symmetric operator and  $\mathcal{A}$  a skew-symmetric operator in a Hilbert space. Then

$$(4) \quad | \langle -[\mathcal{S}, \mathcal{A}]f, f \rangle | \leq 2 \|\mathcal{S}f\| \|\mathcal{A}f\|.$$

Moreover, the equality is attained when  $\alpha \mathcal{S}f + \mathcal{A}f = 0$  for  $0 \neq \alpha \in \mathbb{R}$ .

To prove Heisenberg uncertainty principle we set

$$\mathcal{S}f = xf, \quad \mathcal{A}f = \nabla f,$$

so we are going to discretize these operators  $\mathcal{S}$  and  $\mathcal{A}$ . We discretize  $\mathbb{R}^d$  with the same step  $h > 0$  in all directions, that is, we consider the discretization nodes  $x_k = kh$  for  $k = (k_1, \dots, k_d) \in \mathbb{Z}^d$ , and we are going to work in the space

$$H_h(\mathbb{Z}^d) = \{(u_k)_{k \in \mathbb{Z}^d} : h^d \sum_{k \in \mathbb{Z}^d} |u_k|^2 + h^d \sum_{k \in \mathbb{Z}^d} |khu_k|^2 < +\infty\}.$$

In the same way we define the space

$$\ell_h^2(\mathbb{Z}^d) = \{(u_k)_{k \in \mathbb{Z}^d} : \|u\|_2^2 = h^d \sum_{k \in \mathbb{Z}^d} |u_k|^2 < +\infty\}.$$

Now we define our versions of the position and momentum operators

$$\mathcal{S}_h u_k = kh u_k = (k_1 h, \dots, k_d h), \quad \mathcal{A}_h u_k = \left( \frac{u_{k+e_1} - u_{k-e_1}}{2h}, \dots, \frac{u_{k+e_d} - u_{k-e_d}}{2h} \right),$$

where  $e_j = (0, \dots, 0, \overbrace{1}^j, 0, \dots, 0)$ , for  $j = 1, \dots, d$ .

It is easy to check that the operators  $\mathcal{S}_h$  and  $\mathcal{A}_h$  are symmetric and skew-symmetric respectively (with the inner product  $\langle u, v \rangle = h^d \sum_{k \in \mathbb{Z}^d} u_k \overline{v_k}$ ). Using (4), we have the following discrete

version of the uncertainty principle:  $\forall u \in H_h(\mathbb{Z}^d)$ ,

$$(5) \quad \left| h^d \sum_{k \in \mathbb{Z}^d} \sum_{j=1}^d \frac{u_{k+e_j} + u_{k-e_j}}{2} \overline{u_k} \right| \leq 2 \left( h^d \sum_{k \in \mathbb{Z}^d} |k h u_k|^2 \right)^{1/2} \left( h^d \sum_{k \in \mathbb{Z}^d} \sum_{j=1}^d \left| \frac{u_{k+e_j} - u_{k-e_j}}{2h} \right|^2 \right)^{1/2}.$$

We can manipulate the left-hand side of (5) to obtain

$$(6) \quad \left| d h^d \sum_{k \in \mathbb{Z}^d} |u_k|^2 - \frac{h^2}{2} h^d \sum_{k \in \mathbb{Z}^d} \sum_{j=1}^d \left| \frac{u_{k+e_j} - u_{k-e_j}}{h} \right|^2 \right| \leq 2 \left( h^d \sum_{k \in \mathbb{Z}^d} |k h u_k|^2 \right)^{1/2} \left( h^d \sum_{k \in \mathbb{Z}^d} \sum_{j=1}^d \left| \frac{u_{k+e_j} - u_{k-e_j}}{2h} \right|^2 \right)^{1/2}.$$

In order to take the continuous limit we consider that  $u = (u_k)_{k \in \mathbb{Z}^d}$  is the discretization of a function  $f(x) \in \mathcal{S}(\mathbb{R}^d)$  (in other words,  $u_k = f(x_k) = f(kh)$  for some  $f$ ) and we let  $h$  tend to zero. We notice that the second sum in the left-hand side tends to zero when  $h$  tends to zero. Indeed, without the factor  $h^2/2$  this sum would tend to  $\int_{\mathbb{R}^d} |\nabla f(x)|^2 dx$ , since we have the forward finite difference operator of first order. Therefore, adding the factor  $h^2/2$  makes the sum tend to zero. The other sums tend to their respective integrals in the classic Heisenberg uncertainty principle (1).

Now we will rewrite this inequality in the Fourier space. If we look at  $u_k$  as the Fourier coefficient of a  $\frac{2\pi}{h}$ -periodic function in each variable  $f$ , we have the following relations between  $u$  and  $f$ ,

$$(7) \quad \begin{aligned} u_k &= \hat{f}(k) = \int_{[-\pi/h, \pi/h]^d} f(\xi) e^{-i\xi \cdot kh} d\xi, \\ f(x) &= \left( \frac{h}{2\pi} \right)^d \sum_{k \in \mathbb{Z}^d} u_k e^{i h k \cdot x}. \end{aligned}$$

Considering these relations, we can rewrite the inequality (6) to have

$$(8) \quad \left| \int_{[-\pi/h, \pi/h]^d} \sum_{j=1}^d \cos(x_j h) |f(x)|^2 dx \right| \leq 2 \left( \int_{[-\pi/h, \pi/h]^d} |\nabla f(x)|^2 dx \right)^{1/2} \left( \int_{[-\pi/h, \pi/h]^d} \sum_{j=1}^d \left| \frac{\sin(x_j h)}{h} f(x) \right|^2 dx \right)^{1/2}.$$

As it was pointed out in [11], we have to exclude some cases in (6). If we want to give an inequality of the type  $ab \geq 1$ , we need to normalize one quantity that can be zero, so we assume

that the function  $f$  verifies

$$(9) \quad \sum_{j=1}^d \int_{[-\pi/h, \pi/h]^d} \cos(x_j h) |f|^2 dx \neq 0,$$

and, under this assumption we can normalize (8).

In the sequence space, this condition means that we have to work with sequences such that

$$\Re h^d \sum_{k \in \mathbb{Z}^d} \sum_{j=1}^d u_k \overline{u_{k+e_j}} \neq 0,$$

but it is easy to see that the subspace of this sequences is dense in  $\ell_h^2(\mathbb{Z}^d)$ . If we are given an  $\epsilon > 0$  and  $0 \neq u \in \ell_h^2(\mathbb{Z}^d)$ , then adding  $c_h \epsilon$  in the appropriate coordinate gives us an  $\omega$  such that  $\|u - \omega\|_2^2 = h^d \sum_{k \in \mathbb{Z}^d} |u_k - \omega_k|^2 \leq \epsilon$  and  $\Re h^d \sum_{k \in \mathbb{Z}^d} \omega_k \overline{\omega_{k+e_j}} \neq 0$ .

Once we have this uncertainty relation, we are interested in knowing for which sequence the equality in (6) holds. This sequence  $\omega$  has to verify, for  $0 \neq \alpha \in \mathbb{R}$ , the equation  $\alpha \mathcal{S}_h \omega + \mathcal{A}_h \omega = \mathbf{0}$ , where  $\mathbf{0}$  is the sequence whose components are all zero. Then, we have the recurrence relation

$$\alpha \mathcal{S}_h \omega_k + \mathcal{A}_h \omega_k = 0, \quad \forall k \in \mathbb{Z}^d \iff \alpha k_j h \omega_k + \frac{\omega_{k+e_j} - \omega_{k-e_j}}{2h}, \quad \forall k \in \mathbb{Z}^d, j = 1, \dots, d.$$

This is the recurrence relation satisfied by a product of modified Bessel functions of the first and second kind. However, we will use the Fourier method to find the minimizing sequence, because the uncertainty principle in the Fourier space is also interesting. If we solve the recurrence looking at  $\omega_k$  as the Fourier coefficient of a  $\frac{2\pi}{h}$ -periodic function in each variable  $f(x)$ , we have

$$(10) \quad (\alpha \mathcal{S}_h + \mathcal{A}_h) \omega_k = 0, \forall k \in \mathbb{Z}^d \iff \alpha \partial_{x_j} f(x) + \frac{\sin(x_j h)}{h} f(x) = 0, \quad j = 1, \dots, d.$$

Solving the equation, we get

$$f(x) = C e^{\sum_{j=1}^d \frac{\cos(x_j h)}{\alpha h^2}}.$$

We set the constant  $C = C_{\alpha, h}$ , for example, in order to make the norm in the  $L^2[-\pi/h, \pi/h]^d$  space of  $f$  equal to 1. We can also set the constant  $C$  taking into account the normalization condition (9), to make that quantity equal to 1.

*Remark 2.1.* In this periodic case, this function is in the appropriate Hilbert space  $\forall \alpha \neq 0$ , while in the continuous case it is required the extra condition  $\alpha > 0$ . To study convergence to the classic case we will assume that  $\alpha > 0$ .

Once we know who  $f(x)$  is, we compute the  $k$ -th Fourier coefficient of  $f(x)$  to get  $\omega_k$ ,

$$\omega_k = \int_{[-\pi/h, \pi/h]^d} f(x) e^{-ix \cdot kh} dx.$$

As we have said above, this coefficient is related to the modified Bessel function of the first kind, which has lots of representations, such as

$$I_m(z) = \frac{1}{\pi} \int_0^\pi e^{z \cos \theta} \cos(m\theta) d\theta.$$

Then, it is easy to check that, in order to have  $\|f\|_{L^2[-\pi/h, \pi/h]^d} = 1$ ,

$$C_{\alpha, h} = \left( \frac{h}{2\pi I_0\left(\frac{2}{\alpha h^2}\right)} \right)^{d/2}, \quad \omega_k = C_{\alpha, h} \left( \frac{2\pi}{h} \right)^d \prod_{j=1}^d I_{k_j} \left( \frac{1}{\alpha h^2} \right).$$

The modified Bessel function of the second kind  $K_{k_j}(1/\alpha h^2)$  also verifies this recurrence relation (if we multiply it by the factor  $(-1)^{k_j}$ ), but this sequence is not in  $\ell_h^2(\mathbb{Z}^d)$ , so it makes no sense to consider this sequence, and this is the reason why we only get the modified Bessel function of the first kind using the Fourier method.

Now we will see that the continuous limit of our minimizing sequence is the Gaussian  $e^{-\alpha|x|^2/2}$ , so we recover the minimizer of the Heisenberg uncertainty principle. We shall use the following asymptotic formula for the modified Bessel function of the first kind (see [12] p.203)

$$(11) \quad I_{k_j}(t) = \frac{e^t}{\sqrt{2\pi t}} \sum_{m=0}^{\infty} \frac{(-1)^m (k_j, m)}{(2t)^m}, \quad (t \rightarrow \infty),$$

with  $(k_j, 0) = 1$  and

$$(k_j, m) = \frac{(4k_j^2 - 1)(4k_j^2 - 9) \cdots (4k_j^2 - (2m - 1)^2)}{2^{2m} m!}, \quad m \neq 0.$$

In our case,  $t = 1/\alpha h^2$  tends to  $+\infty$  when  $h$  tends to zero. Therefore, using this formula, since we are considering that  $k_j h = x_j$ , we have

$$I_0 \left( \frac{2}{\alpha h^2} \right) = \frac{\sqrt{\alpha} h e^{2/\alpha h^2}}{2\sqrt{\pi}} (1 + O(h^2)),$$

$$I_{k_j} \left( \frac{1}{\alpha h^2} \right) = \frac{\sqrt{\alpha} h e^{1/\alpha h^2}}{\sqrt{2\pi}} (e^{-\alpha x_j^2/2} + O(h^2)).$$

Hence,

$$\omega_k = \left( \frac{2\pi}{h} \right)^d \left( \frac{2h\sqrt{\pi}}{2\pi h \sqrt{\alpha} e^{2/\alpha h^2} (1 + O(h^2))} \right)^{d/2} \frac{\alpha^{d/2} h^d e^{d/\alpha h^2}}{(2\pi)^{d/2}} (e^{-\alpha|x|^2/2} + O(h^2))$$

$$\xrightarrow{h \rightarrow 0} (4\alpha\pi)^{d/4} e^{-\alpha|x|^2/2},$$

and we have recovered the minimizer of the Heisenberg uncertainty principle.

*Remark 2.2.* The number  $(4\alpha\pi)^{d/4}$  appears because we have asked  $f$  to have norm 1 in the space  $L^2[-\pi/h, \pi/h]^d$ , so the  $\ell_h^2(\mathbb{Z}^d)$  norm of the sequence  $\omega$  is, according to (7),  $(2\pi)^d$  and then the continuous limit has to have the same norm (now in  $L^2(\mathbb{R}^d)$ ). If we study the convergence of  $f(x)$ , it converges to  $(\pi\alpha)^{-d/4} e^{-|x|^2/2\alpha}$ , which is the Fourier transform of the limit of the sequence and it has norm 1.

This uncertainty principle (6) is not new, as we have pointed out in the introduction. In [2], the authors used the inequality we have used here (in one dimension and in the Fourier space). Since they also considered the position operator given by  $\cos(x)$ , they presented another

uncertainty relation in their paper. In order to get convergence, we put the uncertainty relation in the following way

$$(12) \quad 2 \left( \int_{-\pi/h}^{\pi/h} |f'|^2 \right)^{1/2} \left( \int_{-\pi/h}^{\pi/h} |\cos(xh)f|^2 \right)^{1/2} \geq \left| h \int_{-\pi/h}^{\pi/h} \sin(xh)|f|^2 \right|.$$

Although in this case, when  $h$  tends to zero this relation does not converge to any uncertainty relation since the right-hand side goes to zero, the study of this relation in the discrete setting can be interesting.

The discrete operators which give this inequality are

$$\widetilde{\mathcal{S}}_h u_k = kh u_k, \quad \widetilde{\mathcal{A}}_h u_k = i \frac{u_{k+1} + u_{k-1}}{2}.$$

Notice that we multiply by  $i$  so that  $\widetilde{\mathcal{A}}_h$  is skew-symmetric and now  $d = 1$  so  $k$  is a *number*, not a *tuple*. The continuous versions of these operators are

$$\widetilde{\mathcal{S}}f = xf, \quad \widetilde{\mathcal{A}}f = if,$$

and we see here that in the continuous setting we do not have an analogous uncertainty relation since these operators commute and we would have the relation

$$2 \left( \int_{\mathbb{R}} |xf|^2 \right)^{1/2} \left( \int_{\mathbb{R}} |f|^2 \right)^{1/2} \geq 0.$$

If we calculate the minimizing sequence, it corresponds in Fourier with the periodic function  $g(x) = C e^{\sin(xh)/\alpha h}$  and in the sequence space with  $\omega_k = C_{\alpha,h} i^{-k} I_k \left( \frac{1}{\alpha h} \right)$ , so we have almost the same sequence (forgetting the  $h$ ) we had before. In Fourier, it is quite easy to check that the minimizing function goes to zero when  $h$  goes to zero. It makes sense to have the same minimizing sequence with a factor  $i^k$  since we can go (assume for a moment  $h = 1$  and  $d = 1$ ) from (8) to (12) by doing the change of variables  $y = x - \pi/2$  which gives the factor  $i^k$  in the sequence space.

The difference between the two cases is that in the first case the minimizing's center was fixed, but now it depends on  $h$ , as we can see in Figure 1. Moreover, in the first case the value at the center of the Gaussian goes to a constant, but in the second case it goes to zero.

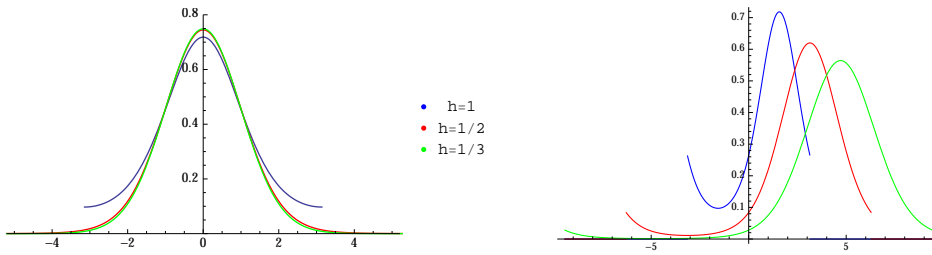


FIGURE 1. Minimizing function  $f(x)$  (left) and  $g(x)$  (right)

In this case the discrete uncertainty relation is

$$(13) \quad \left| h^2 \sum_{k=-\infty}^{\infty} \frac{u_{k+1} - u_{k-1}}{2} u_k \right| \leq 2 \left( h \sum_{k=-\infty}^{\infty} |k h u_k|^2 \right)^{1/2} \left( h \sum_{k=-\infty}^{\infty} \left| \frac{u_{k+1} + u_{k-1}}{2} \right|^2 \right)^{1/2}.$$

### 3. VIRIAL IDENTITY

In this section we are going to give a **discrete Virial identity** equivalent to (2), which relates evolution equations to the inequalities (6) and (13). Then we will use it to obtain a dynamic uncertainty principle equivalent to (3).

First of all we define the discrete Laplacian as the composition of the backward and the forward difference operators, that is,

$$\Delta_d u_k = \sum_{j=1}^d \frac{u_{k+e_j} - 2u_k + u_{k-e_j}}{h^2} = \sum_{j=1}^d \partial_j^+ \partial_j^- u_k,$$

where

$$\partial_j^+ u_k = \frac{u_{k+e_j} - u_k}{h} \Rightarrow \partial_j^- u_k = \frac{u_k - u_{k-e_j}}{h}.$$

Notice that  $(\partial_j^+)^* = -\partial_j^-$ .

We have the following result, equivalent to (2),(3):

**Theorem 3.1** (Dynamic discrete uncertainty principle). *Assume  $u = (u_k(t))_k$  is a solution to the discrete Schrödinger equation*

$$\begin{cases} \partial_t u_k = i \Delta_d u_k, & k \in \mathbb{Z}^d, t \in \mathbb{R}, \\ u_k(0) = u_k^0, \end{cases}$$

where  $u^0 = (u_k^0)_k \in H_h(\mathbb{Z}^d)$  such that

$$(14) \quad h^d \sum_{k \in \mathbb{Z}^d} \sum_{j=1}^d u_k^0 \overline{u_{k+e_j}^0} = 2.$$

For a real  $\phi = (\phi_k)_{k \in \mathbb{Z}^d}$  we define

$$F(t) = h^d \sum_{k \in \mathbb{Z}^d} \phi_k |u_k(t)|^2, \quad a = h^d \sum_{k \in \mathbb{Z}^d} |k h u_k^0|^2 < +\infty, \quad b = h^d \sum_{k \in \mathbb{Z}^d} \sum_{j=1}^d \left| \frac{u_{k+e_j}^0 - u_{k-e_j}^0}{2h} \right|^2 < +\infty.$$

Then, if  $\phi_k$  is of the form  $\phi_{k_1} + \dots + \phi_{k_d}$ ,

$$(15) \quad \ddot{F}(t) = 4h^d \sum_{k \in \mathbb{Z}^d} D^2 \phi_k \mathcal{A}_h u_k \overline{\mathcal{A}_h u_k} - h^d \sum_{k \in \mathbb{Z}^d} \Delta_d^2 \phi_k |u_k|^2,$$

where

$$D^2 \phi_k = \begin{pmatrix} \partial_1^+ \partial_1^- \phi_k & & 0 \\ & \ddots & \\ 0 & & \partial_d^+ \partial_d^- \phi_k \end{pmatrix}.$$



Moreover, if  $\phi_k = (k_1^2 + \dots + k_d^2)h^2$ ,

$$(16) \quad F(t) = a + 4bt^2 \geq a + \frac{4t^2}{a},$$

and, if these two convex parabolas intersect each other, they are the same parabola and the initial datum is  $u_k^0 = \omega_k = C_{\alpha,h} I_k \left( \frac{1}{\alpha h^2} \right)$ , being the solution

$$(17) \quad u_k(t) = e^{it\Delta_d} \omega_k = \omega_k(t) = e^{-\frac{2dit}{h^2}} C_{\alpha,h} \prod_{j=1}^d I_{k_j} \left( \frac{1 + 2\alpha it}{\alpha h^2} \right).$$

*Remark 3.1.* Observe that the Hessian is a diagonal matrix since  $\partial_j^\pm \partial_l^\pm \phi_k = 0$  for  $j \neq l$ . If we do not make this extra assumption on  $\phi$ , we get some extra terms in the expression of  $\ddot{F}(t)$ .

*Remark 3.2.* Notice that now  $C_{\alpha,h}$  changes because of the normalization condition.

*Proof.* For convenience, we are going to use the notation that we have used in the continuous case. That is, we say that  $u$  verifies  $\partial_t u = i \sum_{j=1}^d \partial_j^+ \partial_j^- u$  and we denote  $F(t) = \int \phi(x) |u(x,t)|^2 dx$ .

We will need a discrete Leibniz rule, and there are many ways to write this discrete rule. For example,

$$\partial_j^+ (\phi_k u_k) = \frac{\phi_{k+e_j} u_{k+e_j} - \phi_k u_k}{h} = \frac{\phi_{k+e_j} - \phi_k}{h} u_k + \frac{u_{k+e_j} - u_k}{h} \phi_k + h \frac{u_{k+e_j} - u_k}{h} \frac{\phi_{k+e_j} - \phi_k}{h},$$

which we denote  $\partial_j^+ (\phi u) = u \partial_j^+ \phi + \phi \partial_j^+ u + h \partial_j^+ u \partial_j^+ \phi$ . In the same way, we have another Leibniz rule for  $\partial_j^-$ ,  $\partial_j^- (\phi u) = u \partial_j^- \phi + \phi \partial_j^- u - h \partial_j^- u \partial_j^- \phi$ . Moreover, we have

$$(18) \quad \begin{aligned} \partial_j^+ u + \partial_j^- u &= 2\partial_j^s u \text{ (symmetric difference operator),} \\ \partial_j^+ u - \partial_j^- u &= h \partial_j^+ \partial_j^- u. \end{aligned}$$

Then, taking a time derivative we have, formally

$$\dot{F}(t) = 2\Re \int \phi u \overline{\partial_t u} = 2\Im \sum_{j=1}^d \int \phi u \overline{\partial_j^+ \partial_j^- u} = -2\Im \sum_{j=1}^d \int \partial_j^+ (\phi u) \overline{\partial_j^+ u} = -2\Im \sum_{j=1}^d \int \partial_j^+ \phi u \overline{\partial_j^+ u}.$$

Taking another derivative and using the assumption on  $\phi_k$ , the Leibniz rule and (18),

$$\begin{aligned} \ddot{F}(t) &= 4\Re \sum_{j=1}^d \int \partial_j^+ \partial_j^- \phi \partial_j^s u \overline{\partial_j^+ u} - \sum_{j=1}^d \int (\partial_j^+ \partial_j^-)^2 \phi |u|^2 + h \sum_{j=1}^d \int \partial_j^+ \partial_j^- \partial_j^+ \phi \partial_j^+ \partial_j^- u \overline{u} \\ &\quad + 2h \sum_{j=1}^d \int \partial_j^+ \partial_j^- \partial_j^+ \phi |\partial_j^+ u|^2 + h \sum_{j=1}^d \int (\partial_j^+ \partial_j^-)^2 \phi \partial_j^- u \overline{u}. \end{aligned}$$

On the other hand, following the same procedure, but interchanging the role of  $\partial_j^+$  and  $\partial_j^-$ , we get

$$\begin{aligned} \ddot{F}(t) &= 4\Re \sum_{j=1}^d \int \partial_j^+ \partial_j^- \phi \partial_j^s u \overline{\partial_j^- u} - \sum_{j=1}^d \int (\partial_j^+ \partial_j^-)^2 \phi |u|^2 - h \sum_{j=1}^d \int \partial_j^- \partial_j^+ \partial_j^- \phi \partial_j^+ \partial_j^- u \overline{u} \\ &\quad - 2h \sum_{j=1}^d \int \partial_j^- \partial_j^+ \partial_j^- \phi |\partial_j^- u|^2 - h \sum_{j=1}^d \int (\partial_j^+ \partial_j^-)^2 \phi \partial_j^+ u \overline{u}. \end{aligned}$$

The sum of these two formulae and (18) gives

$$\begin{aligned} 2\ddot{F}(t) &= 8 \int D^2 \phi \mathcal{A}_h u \overline{\mathcal{A}_h u} - 2 \int \Delta_d^2 \phi |u|^2 \\ &\quad + 2h \sum_{j=1}^d \int \partial_j^+ \partial_j^- \partial_j^+ \phi |\partial_j^+ u|^2 - 2h \sum_{j=1}^d \int \partial_j^- \partial_j^+ \partial_j^- \phi |\partial_j^- u|^2. \end{aligned}$$

Finally, we notice that, for  $j = 1, \dots, d$ ,

$$2h \int \partial_j^+ \partial_j^- \partial_j^+ \phi |\partial_j^+ u|^2 - 2h \int \partial_j^- \partial_j^+ \partial_j^- \phi |\partial_j^- u|^2 = 0.$$

Hence,

$$\ddot{F}(t) = 4h^d \sum_{k \in \mathbb{Z}^d} D^2 \phi_k \mathcal{A}_h u_k \overline{\mathcal{A}_h u_k} - h^d \sum_{k \in \mathbb{Z}^d} \Delta_d^2 \phi_k |u_k|^2.$$

Now, as in the continuous case, we take  $\phi_k = |x_k|^2 = |kh|^2 = h^2(k_1^2 + \dots + k_d^2)$ , the discretization of  $|x|^2$ , and we get the two terms of the right-hand side of (6). Indeed,

$$D^2 \phi_k = 2I_d, \quad \Delta_d^2 \phi_k = 0,$$

where  $I_d$  is the identity matrix of order  $d \times d$ . Then,

$$\begin{aligned} F(t) &= h^d \sum_{k \in \mathbb{Z}^d} |kh u_k(t)|^2, \\ \ddot{F}(t) &= 8h^d \sum_{k \in \mathbb{Z}^d} \sum_{j=1}^d \left| \frac{u_{k+e_j} - u_{k-e_j}}{2h} \right|^2. \end{aligned}$$

Moreover,  $\ddot{F}(t) = 0$ .

We can see this fact looking at our equation in the Fourier space. If we consider  $u_k(t) = \hat{f}(k, t)$ , the equation  $\partial_t u_k = i\Delta_d u_k$  is equivalent to

$$\partial_t f(x, t) = i \sum_{j=1}^d \frac{2 \cos(x_j h) - 2}{h^2} f(x, t) = -4i \sum_{j=1}^d \frac{\sin^2(x_j h/2)}{h^2} f(x, t),$$

whose solution is

$$f(x, t) = e^{-4i \sum_{j=1}^d \frac{\sin^2(x_j h/2)t}{h^2}} f(x, 0).$$

Then it is quite obvious that the  $L^2[-\frac{\pi}{h}, \frac{\pi}{h}]^d$  norm of  $f$  is preserved, and so it is the  $\ell_h^2$  norm of  $u$ . Since  $\left( \frac{u_{k+e_j} - u_{k-e_j}}{2h} \right)_k$  verifies the same equation,  $\ddot{F}(t) = \ddot{F}(0)$ . In the same way, we can see that the normalization condition (14) is also preserved with the time. To make these calculations rigorous we also use the equation in the Fourier space. Then, thanks to the expression of the solution, it is quite easy to check that

$$(19) \quad F(t) = \|\nabla f(t)\|_{L^2[-\pi/h, \pi/h]} \leq C(t\|f(0)\|_{L^2[-\pi/h, \pi/h]} + \|\nabla f(0)\|_{L^2[-\pi/h, \pi/h]}) < +\infty,$$

since  $u^0$  is in  $H_h(\mathbb{Z}^d)$ . In fact, we can refine these estimate of  $F(t)$  to give it in terms of  $a$  and  $b$ , but, since we are working on discrete spaces,  $b$  is controlled by  $\|u^0\|_2$ .

These facts allow us to write  $F(t)$  as a convex parabola, and, we can assume without loss of generality  $\dot{F}(0) = 0$  (if not, we make a translation in time). Then  $F(t) = a + 4bt^2$ . Furthermore, by (6) and (14), the coefficients of this parabola verify the inequality

$$\sqrt{ab} \geq 1,$$

so (16) holds.

As in the continuous case, if the equality holds, then we know that for some  $\alpha$ ,  $(u_k^0)_k = (\omega_k)_k$  is the minimizing sequence. If we solve the equation

$$\begin{cases} \partial_t u_k(t) = i\Delta_d u_k(t), & k \in \mathbb{Z}^d, t \in \mathbb{R}, \\ u_k(0) = \omega_k = C_{\alpha,h} \prod_{j=1}^d I_{k_j} \left( \frac{1}{\alpha h^2} \right), \end{cases}$$

we get, by properties of the modified Bessel functions (see [12])

$$\begin{aligned} e^{it\Delta_d} \omega_k &= u_k(t) = e^{-\frac{2dit}{h^2}} C_{\alpha,h} \prod_{j=1}^d \sum_{m_j \in \mathbb{Z}} I_{m_j} \left( \frac{1}{\alpha h^2} \right) I_{k_j - m_j} \left( \frac{2it}{h^2} \right) \\ &= e^{-\frac{2dit}{h^2}} C_{\alpha,h} \prod_{j=1}^d I_{k_j} \left( \frac{1 + 2\alpha it}{\alpha h^2} \right), \end{aligned}$$

hence, the solution has the same form as the initial datum, both are products of modified Bessel functions of the first kind.  $\square$

*Remark 3.3.* For  $\gamma \in \mathbb{R}$ , if we consider the equation

$$\partial_t u_k(t) = i \sum_{j=1}^d \frac{u_{k+e_j} + \gamma u_k + u_{k-e_j}}{h^2},$$

and we repeat all the calculations, we get to the same result. Note that multiplying  $u_k$  by an appropriate exponential term we can reduce this equation to

$$\partial_t v_k(t) = i \sum_{j=1}^d \frac{v_{k+e_j} + v_{k-e_j}}{h^2},$$

so dealing with this equation is enough to see the general case with  $\gamma$ , and, in particular, the discrete Schrödinger equation ( $\gamma = -2$ ).

Since in Fourier we also have the other periodic uncertainty principle, one can think that this new discrete uncertainty relation (13), although it is not a discrete version of Heisenberg uncertainty principle, satisfies another Virial identity, but the natural choice for the equation (the composition of the “backward summation operator” and the “forward summation operator” fails, fact that we have pointed out in the previous remark, since it would be the case  $\gamma = 2$ ).

The question then is: Is there any equation (we restrict ourselves to the one dimensional case)  $\partial_t u_k = \mathcal{T} u_k \ell_h^2(\mathbb{Z})$ -invariant such that

$$F(t) = h \sum_{k=-\infty}^{\infty} |k h u_k(t)|^2 \Rightarrow \ddot{F}(t) = C h \sum_{k=-\infty}^{\infty} \left| \frac{u_{k+1}(t) + u_{k-1}(t)}{2} \right|^2, \quad \ddot{F}(t) = 0 ?$$

We present here two equations that answer the question in an affirmative way. Since these two equations are not very different, we present a general result and later we will talk about the equations in detail.

**Theorem 3.2.** Assume  $u = (u_k(t))_k$  and  $v = (v_k(t))_k$  satisfy

$$\begin{cases} \partial_t u_k = i \frac{v_{k+1} - v_{k-1}}{2h}, & \partial_t v_k = -i \frac{u_{k+1} - u_{k-1}}{2h}, \\ u_k(0) = u_k^0, & v_k(0) = v_k^0, \end{cases}$$

where  $u^0 = (u_k^0)_k$ ,  $v = (v_k^0)_k \in H_h(\mathbb{Z})$  and

$$h^2 \sum_{k=-\infty}^{\infty} \frac{u_{k+1}^0 - u_{k-1}^0}{2} \overline{u_k^0} = 2.$$

We define

$$F(t) = h \sum_{k=-\infty}^{\infty} k^2 h^2 |u_k(t)|^2, \quad a = h \sum_{k=-\infty}^{\infty} k^2 h^2 |u_k^0|^2 < +\infty, \quad b = h \sum_{k=-\infty}^{\infty} \left| \frac{u_{k+1}^0 + u_{k-1}^0}{2} \right|^2 < +\infty.$$

Then, if  $\forall k \in \mathbb{Z}, \forall t, |u_k(t)|^2 = |v_k(t)|^2$  and  $\Re(u_{k+1}(t) \overline{u_{k-1}(t)}) = \Re(v_{k+1}(t) \overline{v_{k-1}(t)})$ ,

$$\ddot{F}(t) = 2h \sum_{k=-\infty}^{\infty} \left| \frac{u_{k+1}(t) + u_{k-1}(t)}{2} \right|^2, \quad \ddot{F}(t) = 0,$$

and the system is  $\ell_h^2(\mathbb{Z})$ -invariant. Moreover,

$$F(t) = a + bt^2 \geq a + \frac{t^2}{a},$$

and, if these two parabolas intersect each other, they are the same parabola and the initial datum  $u^0$  is the minimizer of (13).

*Proof.* To begin with, we will prove the  $\ell_h^2(\mathbb{Z})$ -invariance. If we differentiate the equality given by the hypothesis  $\|u(t)\|_2 = \|v(t)\|_2$ , then we have

$$\Re \sum_{k=-\infty}^{\infty} i(v_{k+1} - v_{k-1}) \overline{u_k} = -\Re \sum_{k=-\infty}^{\infty} i(u_{k+1} - u_{k-1}) \overline{v_k}.$$

Then adding this sums, dividing by 2 and using that  $\Im(z) = -\Im(\bar{z})$ , we have

$$\partial_t \|u(t)\|_2 = \frac{1}{2} \Im \sum_{k=-\infty}^{\infty} (\bar{v}_k u_{k-1} - \bar{v}_k u_{k+1} + u_{k+1} \bar{v}_k - u_{k-1} \bar{v}_k) = 0.$$

Using the same procedure, we observe that

$$\partial_t \Re \sum_{k=-\infty}^{\infty} u_{k+1} \overline{u_{k-1}} = \partial_t \Re \sum_{k=-\infty}^{\infty} v_{k+1} \overline{v_{k-1}} = 0,$$

which will be useful later.

Now,

$$\dot{F}(t) = h^2 \Re \sum_{k=-\infty}^{\infty} k^2 i(v_{k+1} - v_{k-1}) \overline{u_k} = -h^2 \Im \sum_{k=-\infty}^{\infty} k^2 (v_{k+1} \overline{u_k} - v_{k-1} \overline{u_k}).$$

Differentiating again we have

$$\begin{aligned}
\ddot{F}(t) &= -\frac{h}{2} \Re \sum_{k=-\infty}^{\infty} k^2 (-u_{k+2} \overline{u_k} + 2|u_k|^2 - |v_{k+1}|^2 - |v_{k-1}|^2 + 2v_{k+1} \overline{v_{k-1}} - u_{k-2} \overline{u_k}) \\
&= \frac{h}{2} \Re \sum_{k=-\infty}^{\infty} \left( (k-1)^2 (u_{k+1} \overline{u_{k-1}} + |v_k|^2) - 2k^2 (|u_k|^2 + v_{k+1} \overline{v_{k-1}}) \right. \\
&\quad \left. + (k+1)^2 (|v_k|^2 + u_{k+1} \overline{u_{k-1}}) \right) \\
&= h \Re \sum_{k=-\infty}^{\infty} (|u_k|^2 + u_{k+1} \overline{u_{k-1}}) = 2h \sum_{k=-\infty}^{\infty} \left| \frac{u_{k+1} + u_{k-1}}{2} \right|^2.
\end{aligned}$$

Moreover, using the previous calculations it is now obvious that  $\ddot{\ddot{F}}(t) = 0$ . Again, this formal calculations are rigorous if we look at the system in the Fourier space, having a similar estimate to (19). Now, assuming without loss of generality  $\dot{F}(0) = 0$ , and since by (13)

$$\sqrt{ab} \geq 1,$$

we have  $F(t) = a + bt^2 \geq a + \frac{t^2}{a}$ , and if these two parabolas intersect each other, we have the equality in (13) and then  $u^0$  has to be the minimizing sequence.  $\square$

Now we are going to solve the system using the Fourier method. The system that we want to solve is

$$\begin{cases} \partial_t u_k = i \frac{v_{k+1} - v_{k-1}}{2h} & u_k(0) = u_k^0, \\ \partial_t v_k = -i \frac{u_{k+1} - u_{k-1}}{2h} & v_k(0) = v_k^0. \end{cases}$$

We consider  $f_0(x), g_0(x)$   $\frac{2\pi}{h}$ -periodic functions such that

$$u_k^0 = \hat{f}_0(k), \quad v_k^0 = \hat{g}_0(k) \quad \text{and} \quad u_k(t) = (f(t))^\wedge(k), \quad v_k(t) = (g(t))^\wedge(k),$$

so the system in the Fourier space is

$$\begin{cases} \partial_t f = \frac{\sin(xh)}{h} g & f(x, 0) = f_0(x), \\ \partial_t g = -\frac{\sin(xh)}{h} f & g(x, 0) = g_0(x). \end{cases}$$

We have a system of two ODEs, whose solution is

$$\begin{aligned}
f(x, t) &= \frac{f_0(x) - ig_0(x)}{2} e^{i \sin(xh)t/h} + \frac{f_0(x) + ig_0(x)}{2} e^{-i \sin(xh)t/h}, \\
g(x, t) &= \frac{g_0(x) + if_0(x)}{2} e^{i \sin(xh)t/h} + \frac{g_0(x) - if_0(x)}{2} e^{-i \sin(xh)t/h}.
\end{aligned}$$

Finally, we recover from these expressions the value of  $u_k(t)$  and  $v_k(t)$ .

$$u_k(t) = \int_{-\pi/h}^{\pi/h} f(x, t) e^{-ixkh} dx, \quad v_k(t) = \int_{-\pi/h}^{\pi/h} g(x, t) e^{-ixkh} dx.$$

We can prove the  $\ell_h^2(\mathbb{Z})$ -invariance in the Fourier space too, proving that  $f$  and  $g$  are  $L^2[-\frac{\pi}{h}, \frac{\pi}{h}]$ -invariants. We only have to use that  $\|f(t)\|_2 = \|g(t)\|_2 \quad \forall t$ , which is true thanks to the hypothesis on  $u_k$  and  $v_k$ .

The two equations we want to mention here are the cases when we set, on the one hand  $v_k = \overline{u_k}$  and, on the other hand,  $v_k = (-1)^{k+1}u_k$ . It is easy to check that these two options verify the hypothesis of the Theorem 3.2.

**First case**  $v_k = \overline{u_k}$ .

In this case we can state the Virial principle as follows:

**Corollary 3.1.** *Assume  $u = (u_k(t))_k$  satisfies*

$$\partial_t u_k = i \frac{\overline{u_{k+1}} - \overline{u_{k-1}}}{2h}, \quad u_k(0) = u_k^0,$$

where  $u^0 = (u_k^0)_k \in H_h(\mathbb{Z})$  and

$$h^2 \sum_{k=-\infty}^{\infty} \frac{u_{k+1}^0 - u_{k-1}^0}{2} \overline{u_k^0} = 2.$$

We define

$$F(t) = h \sum_{k=-\infty}^{\infty} k^2 h^2 |u_k(t)|^2, \quad a = h \sum_{k=-\infty}^{\infty} k^2 h^2 |u_k^0|^2 < +\infty, \quad b = h \sum_{k=-\infty}^{\infty} \left| \frac{u_{k+1}^0 + u_{k-1}^0}{2} \right|^2 < +\infty.$$

Then,

$$\ddot{F}(t) = 2h \sum_{k=-\infty}^{\infty} \left| \frac{u_{k+1}^0 + u_{k-1}^0}{2} \right|^2,$$

and the equation is  $\ell_h^2(\mathbb{Z})$ -invariant. Moreover,

$$F(t) = a + bt^2 \geq a + \frac{t^2}{a},$$

and, if these two parabolas intersect each other, they are the same parabola and the initial datum  $u^0$  is the minimizer of (13).

In this case, since  $g_0(x) = \overline{f_0(-x)}$ , the solution to the equation is

$$u_k(t) = \int_{-\pi/h}^{\pi/h} f(x, t) e^{-ixkh} dx,$$

where

$$f(x, t) = \frac{f_0(x) - i\overline{f_0(-x)}}{2} e^{i \sin(xh)t/h} + \frac{f_0(x) + i\overline{f_0(-x)}}{2} e^{-i \sin(xh)t/h}.$$

In this case, the equation is a discrete version of the equation

$$\partial_t f = i \overline{\partial_x f}.$$

If we take another time derivative, we get the wave equation  $\partial_t^2 f = \partial_x^2 f$ , so this equation gives an  $L^2$ -invariant factorization of the one dimensional wave equation. Using *d'Alembert's formula* we can see that the solution to this equation is

$$f(x, t) = \frac{1}{2} (f_0(x+t) + f_0(x-t)) + \frac{i}{2} (\overline{f_0(x+t)} - \overline{f_0(x-t)}).$$

If we take another time derivative in our discrete equation, as it can be expected we get a discrete version of the wave equation

$$\partial_t^2 u_k = \frac{u_{k+2} - 2u_k + u_{k-2}}{4h^2}.$$

In the continuous setting, the analogous of this corollary is:

**Proposition 3.1.** *Assume  $f$  satisfies*

$$\partial_t f = i\overline{\partial_x f}, \quad f(x, 0) = f_0(x),$$

*and let  $F(t) = \int_{\mathbb{R}} x^2 |f(x, t)|^2 dx$ , then*

$$\ddot{F}(t) = 2 \int_{\mathbb{R}} |f(x, t)|^2 = \ddot{F}(0).$$

**Second case**  $v_k = (-1)^{k+1} u_k$ .

In this case we can state the Virial principle as follows:

**Corollary 3.2.** *Assume  $u = (u_k(t))_k$  satisfies*

$$\partial_t u_k = i \frac{(-1)^k u_{k+1} + (-1)^{k-1} u_{k-1}}{2h}, \quad u_k(0) = u_k^0,$$

*where  $u^0 = (u_k^0)_k \in H_h(\mathbb{Z})$  and*

$$h^2 \sum_{k=-\infty}^{\infty} \frac{u_{k+1}^0 - u_{k-1}^0}{2} \overline{u_k^0} = 2.$$

*We define*

$$F(t) = h \sum_{k=-\infty}^{\infty} k^2 h^2 |u_k(t)|^2, \quad a = h \sum_{k=-\infty}^{\infty} k^2 h^2 |u_k^0|^2 < +\infty, \quad b = h \sum_{k=-\infty}^{\infty} \left| \frac{u_{k+1}^0 + u_{k-1}^0}{2} \right|^2 < +\infty.$$

*Then,*

$$\ddot{F}(t) = 2h \sum_{k=-\infty}^{\infty} \left| \frac{u_{k+1}^0 + u_{k-1}^0}{2} \right|^2,$$

*and the equation is  $\ell_h^2(\mathbb{Z})$ -invariant. Moreover,*

$$F(t) = a + bt^2 \geq a + \frac{t^2}{a},$$

*and, if these two parabolas intersect each other, they are the same parabola and the initial datum  $u^0$  is the minimizer of (13).*

In this case, since  $g_0(x) = -f_0\left(x + \frac{\pi}{h}\right)$ , the solution to the equation is

$$u_k(t) = \int_{-\pi/h}^{\pi/h} f(x, t) e^{-ixkh} dx,$$

where

$$f(x, t) = \frac{f_0(x) + if_0\left(x + \frac{\pi}{h}\right)}{2} e^{i \sin(xh)t/h} + \frac{f_0(x) - if_0\left(x + \frac{\pi}{h}\right)}{2} e^{-i \sin(xh)t/h}.$$

4. PROPERTIES OF  $e^{it\Delta_d}u^0$ 

Now we will see that the function (see (17))  $\omega_k(t) = e^{it\Delta_d}\omega_k$ , where  $\omega = (\omega_k)_k$  is the minimizing function to (6), and the solution to the Schrödinger equation  $g(x, t)$  with initial datum  $g_0(x) = e^{-\alpha|x|^2/2}$  have got similar properties.

We recall that  $g_0(x)$  verifies the equation  $\alpha x g_0(x) + \nabla g_0(x) = 0$ , which is a sum of a symmetric and a skew-symmetric operator. It is easy to see that then  $g(x, t)$  verifies

$$g = (\alpha\mathcal{S} + \mathcal{A})g = 0, \text{ where } \mathcal{A}u = \nabla u, \quad \mathcal{S}u = xu + 2it\nabla u.$$

Hence, the solution  $g(x, t)$  satisfies another equation with a symmetric and a skew-symmetric operator. Moreover, if we denote  $\Lambda(t)f = xf + 2it\nabla f$ , we can see that

$$\Lambda(t)e^{it\Delta_d}u_0(x) = e^{it\Delta_d}\Lambda(0)u_0(x) \implies \Lambda(t) = e^{it\Delta_d}\Lambda(0)e^{-it\Delta_d} = e^{it\Delta_d}xe^{-it\Delta_d},$$

where  $e^{it\Delta_d}u_0(x)$  stands for the solution to the problem

$$\begin{cases} \partial_t u(x, t) = i\Delta_d u(x, t), & x \in \mathbb{R}^d, \quad t \in \mathbb{R}, \\ u(x, 0) = u_0(x). \end{cases}$$

We have the following result:

**Theorem 4.1.** *Let  $\omega(t) = (\omega_k(t))_k$  be given by (17). Then*

$$(\mathcal{A}_h + \alpha\Lambda_d(t))\omega_k(t) = 0,$$

where  $\mathcal{A}_h$  is skew-symmetric and  $\Lambda_d(t)$  is symmetric and given by

$$\mathcal{A}_h u_k(t) = \left( \frac{u_{k+e_j}(t) - u_{k-e_j}(t)}{2h} \right)_j, \quad \Lambda_d(t)u_k(t) = kh u_k(t) + 2it\mathcal{A}_h u_k(t).$$

Moreover,

$$\Lambda_d(t)e^{it\Delta_d}u_k^0 = e^{it\Delta_d}\Lambda_d(0)u_k^0 \implies \Lambda_d(t) = e^{it\Delta_d}\Lambda_d(0)e^{-it\Delta_d} = e^{it\Delta_d}khe^{-it\Delta_d}.$$

*Proof.* As the skew-symmetric operator in both equations is the same in the continuous case,  $\mathcal{A}u = \nabla u$ , we will compute  $\mathcal{A}_h\omega_k(t)$  and we will see that we get the symmetric operator from there. Using the recurrence of the function  $I_{k_j}(z)$  we have, for  $j = 1, \dots, d$ ,

$$\begin{aligned} \frac{\omega_{k+e_j}(t) - \omega_{k-e_j}(t)}{2h} &= \frac{e^{-2dit/h^2}C_{\alpha,h}}{2h} \left( I_{k_j+1} \left( \frac{1+2\alpha it}{\alpha h^2} \right) - I_{k_j-1} \left( \frac{1+2\alpha it}{\alpha h^2} \right) \right) \prod_{l \neq j} I_{k_l} \left( \frac{1+2\alpha it}{\alpha h^2} \right) \\ &= -\frac{\alpha k_j h e^{-2dit/h^2} C_{\alpha,h}}{1+2\alpha it} \prod_{l=1}^d I_{k_l} \left( \frac{1+2\alpha it}{\alpha h^2} \right) = -\frac{\alpha k_j h}{1+2\alpha it} \omega_k(t). \end{aligned}$$

Hence,  $\omega_k(t)$  satisfies the equation  $(\mathcal{A}_h + \alpha\Lambda_d)\omega_k(t) = 0$ .



Furthermore, using again the recurrence of  $I_k(z)$  we have

$$\begin{aligned}
\Lambda_d(t)e^{it\Delta_d}u_k^0 &= khe^{-\frac{2dit}{h^2}} \sum_{m \in \mathbb{Z}^d} u_m^0 \prod_{l=1}^d I_{k_l-m_l} \left( \frac{2it}{h^2} \right) \\
&+ \left( \frac{it}{h} e^{-\frac{2dit}{h^2}} \sum_{m \in \mathbb{Z}^d} u_m^0 \left( I_{k_j-m_j+1} \left( \frac{2it}{h^2} \right) - I_{k_j-m_j-1} \left( \frac{2it}{h^2} \right) \right) \prod_{l \neq j} I_{k_l-m_l} \left( \frac{2it}{h^2} \right) \right)_j \\
&= \left( e^{-\frac{2dit}{h^2}} \sum_{m \in \mathbb{Z}^d} u_m^0 \left( k_j h - \frac{it}{h} \frac{2(k_j-m_j)h^2}{2it} \right) \prod_{l=1}^d I_{k_l-m_l} \left( \frac{2it}{h^2} \right) \right)_j \\
&= \left( e^{-\frac{2dit}{h^2}} \sum_{m \in \mathbb{Z}^d} m_j h u_m^0 \prod_{l=1}^d I_{k_l-m_l} \left( \frac{2it}{h^2} \right) \right)_j = e^{it\Delta_d} \Lambda_d(0) u_0^k.
\end{aligned}$$

□

## 5. UNCERTAINTY PRINCIPLES FOR FINITE SEQUENCES

In this section we are going to see some uncertainty relations for finite sequences in one dimension  $u = (u_k)_{k=-N}^{k=N}$ . The motivation comes from [8], where the author gives an uncertainty relation for the DFT considering discrete versions of the position and momentum operators, but, using his words, the minimizer does not “*bear much of a connection with the natural of the Gaussian in this context*”. Here, we introduce a slight modification of his operators in order to relate the new minimizer to the Gaussian. The main difference between this approach and the one in [8] is that here we introduce a new parameter which allows us to recover the Gaussian in a limiting process which consists in two steps. First we recover the minimizing function of the periodic uncertainty principle (8), and then, as we have seen in Section 2 we approach the Gaussian when the period of the minimizing function tends to infinity. Moreover, we give two uncertainty relations truncating the operators we have studied in Section 2 and assuming periodic and Dirichlet conditions.

**5.1. The case of the Discrete Fourier Transform.** The operators we propose here are

$$(20) \quad \mathcal{S}_h = \begin{bmatrix} q_{-N} & & 0 \\ & \ddots & \\ 0 & & q_N \end{bmatrix}, \quad \mathcal{A}_h = \frac{1}{2h} \begin{bmatrix} 0 & 1 & 0 & \cdots & -1 \\ -1 & 0 & 1 & \cdots & 0 \\ 0 & -1 & 0 & \ddots & 0 \\ & & \ddots & \ddots & \\ 1 & 0 & \cdots & -1 & 0 \end{bmatrix},$$

where

$$q_k = \frac{(2N+1)h}{2\pi} \sin \left( \frac{2\pi k}{2N+1} \right), \quad k = -N, \dots, N.$$

*Remark 5.1.* In [8], the author considered the coefficients (in this case for sequences  $(u_k)_{k=0}^{k=N}$  and  $h = 1/2$ )

$$\tilde{q}_k = \sin \left( \frac{2\pi k}{N} \right).$$

With this choice of  $\tilde{q}_k$ , the uncertainty principle in [8] has a nice representation for  $\|\mathcal{A}_h u\|_2$  in terms of the DFT, but, as we have said above, there is no relation between the minimizer and the Gaussian.

Then if we consider the DFT of a sequence

$$\hat{u}_k = \frac{1}{\sqrt{2N+1}} \sum_{j=-N}^N u_j e^{-\frac{2\pi i k j}{2N+1}}, \quad k = -N, \dots, N,$$

the uncertainty principle can be written as

$$(21) \quad 2 \left( h \sum_{k=-N}^N q_k^2 |u_k|^2 \right)^{1/2} \left( h \sum_{k=-N}^N \frac{\sin^2 \left( \frac{2\pi k}{2N+1} \right)}{h^2} |\hat{u}_k|^2 \right)^{1/2} \geq |\langle -[\mathcal{S}_h, \mathcal{A}_h] u, u \rangle|,$$

or

$$(22) \quad 2 \left( h \sum_{k=-N}^N q_k^2 |u_k|^2 \right)^{1/2} \left( h \sum_{k=-N}^N \left( \frac{2\pi}{(2N+1)h^2} \right)^2 q_k^2 |\hat{u}_k|^2 \right)^{1/2} \geq |\langle -[\mathcal{S}_h, \mathcal{A}_h] u, u \rangle|.$$

As we know, the minimizer verifies the relation  $(\mathcal{S}_h + \alpha \mathcal{A}_h) \omega = 0$ , for  $\alpha \neq 0$ . Here we will assume that  $\alpha = 1$  and the initial condition  $\omega_0^h = 1$ . Now we study the behavior of the minimizing sequence of (21) when we let  $h$  tend to zero and  $N$  tend to infinity under the constraints  $Nh = L$  and  $kh = x$ , for  $x \in [-L, L]$ , in order to obtain the continuous limit of the minimizing sequence when we discretize the interval  $[-L, L]$ .

From (21) and (24) we know that the  $k$ -th component of the minimizing sequence satisfies the system, for  $k = -N, \dots, N$

$$\frac{2L+h}{2\pi} \sin \left( \frac{2\pi kh}{2L+h} \right) \omega_k^h + \frac{\omega_{k+1}^h - \omega_{k-1}^h}{2h} = 0, \quad \omega_{N+1}^h = \omega_{-N}^h, \quad \omega_{-N-1}^h = \omega_N^h.$$

This is a discrete version of the equation, for  $x \in [-L, L]$

$$\frac{L}{\pi} \sin \left( \frac{\pi x}{L} \right) \omega(x) + \omega'(x) = 0.$$

Therefore, we should have that the continuous limit of the sequence should be the minimizing function of the periodic uncertainty principle shown in Section 2 (8) and (10), now with the initial condition  $\omega(0) = 1$ , and the role of  $h$  played by the quantity  $\frac{\pi}{L}$ . Hence, as we have shown in Section 2, if we let  $L$  tend to  $\infty$ , then we recover the Gaussian. In Figure 2, we can see how the minimizing sequence approaches the minimizing function of the periodic uncertainty principle.

In order to see the convergence of the minimizer, we consider that  $L = \pi$  and we slightly change  $q_k$  to  $q_k = \sin(kh)$ , and the general case follows directly from this case. Now, we point out that, since  $q_0 = 0$ , we have, by induction  $\omega_k^h = \omega_{-k}^h$ , so from now on we will only have in mind  $k$  positive. Moreover, this fact allows us to construct the solution to the system by an iterative process starting from  $\omega_N^h$  to  $\omega_1^h$ . We have then that

$$\omega_k^h = \frac{1}{[2hq_k, \dots, 2hq_{N-1}, 1 + 2hq_N]} \frac{1}{[2hq_{k-1}, \dots, 1 + 2hq_N]} \cdots \frac{1}{[2hq_1, \dots, 1 + 2hq_N]},$$

where

$$[a_0, a_1, \dots, a_n] = a_0 + \frac{1}{a_1 + \frac{1}{\ddots + \frac{1}{a_n}}}.$$

To deal with this product of continued fractions, we use Theorem 149 in [9], which states that the continued fraction  $[a_0, a_1, \dots, a_r]$  is a rational number  $\frac{p_r}{q_r}$ , where  $p_r$  and  $q_r$  are given by the recurrence

$$\begin{aligned} p_0 &= a_0, \quad p_1 = a_1 a_0 + 1, \quad p_n = a_n p_{n-1} + p_{n-2} \quad (2 \leq n \leq r), \\ q_0 &= 1, \quad q_1 = a_1, \quad q_n = a_n q_{n-1} + q_{n-2} \quad (2 \leq n \leq r). \end{aligned}$$

Hence,  $\omega_k^h = \frac{s_N}{t_N}$ , where

$$\begin{aligned} s_N &= \begin{pmatrix} 1 + 2hq_N & 1 \\ 1 & 0 \end{pmatrix} \prod_{m=N-1}^{k+2} \begin{pmatrix} 2hq_m & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2hq_{k+1} \\ 1 \end{pmatrix}, \\ t_N &= \begin{pmatrix} 1 + 2hq_N & 1 \\ 1 & 0 \end{pmatrix} \prod_{m=N-1}^3 \begin{pmatrix} 2hq_m & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 4h^2 q_1 q_2 + 1 \\ 2hq_1 \end{pmatrix}. \end{aligned}$$

*Remark 5.2.* The notation  $\prod_{m=N-1}^{k+2} \begin{pmatrix} 2hq_m & 1 \\ 1 & 0 \end{pmatrix}$  represents that the first matrix is the one with index  $m = N - 1$ , the following matrix is the one with index  $m = N - 2$ , and so on.

We should distinguish different cases depending on the parity of  $k$  and  $N$ . Here we will assume that  $N, k \equiv 0 \pmod{4}$ . The other cases follow a similar reasoning. Moreover, we will treat the numerator and the denominator separately. Now, if we expand the matrix product of the numerator, we notice that it generates a polynomial expression of  $h$  where the coefficients are given by products of  $q_j$ . To be exact, we have

$$\begin{aligned} s_N &= \sum_{j=0}^{N-k} a_j^h, \quad \text{where} \\ a_0^h &= 1, \quad a_j^h = \sum_{l_1=k/2+1}^{\lfloor \frac{N-j}{2} \rfloor + 1} \sum_{l_2=l_1}^{\lfloor \frac{N-j}{2} \rfloor + 1} \cdots \sum_{l_j=l_{j-1}}^{\lfloor \frac{N-j}{2} \rfloor + 1} q_{2l_1-1} q_{2l_2} \cdots q_{2l_j+j-2}, \quad 1 \leq j \leq N-k, \end{aligned}$$

where  $\lfloor n \rfloor$  stands for the integer part of  $n$ .

In order to study the limit of  $s_N$  when  $h$  tends to zero assuming  $kh = x$ , ( $x \in (0, \pi)$ ) and  $Nh = \pi$ , we notice that each  $a_j^h$ , for  $j$  fixed, is related to a Riemann sum in the interval  $(x/2, \pi/2)$ . Thus,  $a_j^h$  will converge to an iterated integral.

To understand this limit process consider the case  $j = 1$ ,

$$2h \sum_{l_1=\frac{k}{2}+1}^{N/2} q_{2l_1-1} = 2h \sum_{l_1=\frac{k}{2}+1}^{N/2} \sin((2l_1-1)h).$$

We have a uniform partition (whose step is  $h$ ) of the interval  $[(\frac{k}{2}+1)h, \frac{N}{2}h] \xrightarrow{h \rightarrow 0} [\frac{x}{2}, \frac{\pi}{2}]$  of the function

$$f(z) = 2 \sin(2z).$$

Hence,

$$2h \sum_{l_1=\frac{k}{2}+1}^{N/2} q_{2l_1-1} \xrightarrow[h \rightarrow 0]{\substack{Nh=\pi \\ kh=x}} \int_{x/2}^{\pi/2} f(z) dz.$$

In the general case, we can follow the same argument to see that

$$a_j^h \xrightarrow[h \rightarrow 0]{\substack{Nh=\pi \\ kh=x}} 2^j \int_{x/2}^{\pi/2} \int_{x_1}^{\pi/2} \dots \int_{x_{j-1}}^{\pi/2} \sin(2x_1) \sin(2x_2) \dots \sin(2x_j) dx_j \dots dx_2 dx_1,$$

and it is quite easy to check that

$$\begin{aligned} 2^j \int_{x/2}^{\pi/2} \int_{x_1}^{\pi/2} \dots \int_{x_{j-1}}^{\pi/2} \sin(2x_1) \sin(2x_2) \dots \sin(2x_j) dx_j \dots dx_2 dx_1 \\ = \left( 2 \int_{x/2}^{\pi/2} \sin(2z) dz \right)^j \frac{1}{j!} = \frac{(1 + \cos(x))^j}{j!}. \end{aligned}$$

Once we have shown this limit, we are going to bound all the sine functions by 1 to get estimates independent of  $h$ . By an inductive process we can see that

$$\sum_{j=0}^{N-k} a_j^h \leq \sum_{j=0}^{N-k} (2h)^j \binom{\frac{N-k}{2} + \lfloor \frac{j}{2} \rfloor}{j} \leq \left( \sum_{j=0}^{\frac{N-k}{2}} + \sum_{j=\frac{N-k}{2}}^{N-k} \right) (2h)^j \binom{\frac{N-k}{2} + \lfloor \frac{j}{2} \rfloor}{j} = I + II.$$

We will see that we can apply the dominated convergence theorem to  $I$ , while we can make  $II$  as small as we want when  $h$  is small enough. To begin with, we study  $II$  and we point out that these binomial coefficients form two decreasing sequences, one is generated by the case  $j$  even and the other one by the case  $j$  odd. Indeed, if  $j = 2m$  is even,

$$\binom{\frac{N-k}{2} + m}{2m} \geq \binom{\frac{N-k}{2} + m + 1}{2m + 2} \Leftrightarrow 5m^2 + 7m + 2 \geq \left( \frac{N-k}{2} \right)^2 + \frac{N-k}{2},$$

which is true because  $2m \geq \frac{N-k}{2}$ . On the other hand, if  $j = 2m + 1$  is odd, then

$$\binom{\frac{N-k}{2} + m}{2m + 1} \geq \binom{\frac{N-k}{2} + m + 1}{2m + 3} \Leftrightarrow 5m^2 + 12m + 7 \geq \left( \frac{N-k}{2} \right)^2,$$

which is true as well. Moreover, it is quite obvious to check that (recall that  $N, k \equiv 0 \pmod{4}$ , so  $\frac{N-k}{2}$  is even)

$$\binom{\frac{N-k}{2} + \frac{N-k}{4}}{\frac{N-k}{2}} \geq \binom{\frac{N-k}{2} + \frac{N-k}{4}}{\frac{N-k}{2} + 1}.$$

Therefore, we have that  $a_j^h \leq (2h)^{\frac{N-k}{2}} \left( 3^{\frac{N-k}{4}} \right)$ ,  $\forall j \geq \frac{N-k}{2}$ . Observe that we can improve this estimate since in this way we are decreasing the power of  $h$  in each  $a_j^h$  to the power of  $a_{\frac{N-k}{2}}^h$ , but this is enough to prove the convergence. The last bound allows us to say that

$$\sum_{j=\frac{N-k}{2}}^{N-k} a_j^h \leq (2h)^{\frac{N-k}{2}} \left( 3^{\frac{N-k}{4}} \right) \frac{N-k}{2}.$$

The next step consists in proving that this number tends to zero when  $h$  tends to zero. To prove that, we recall that  $Nh = \pi$  and  $kh = x$ , so we write  $N - k = yh^{-1}$ , where  $y = \pi - x > 0$ . Hence, by the Stirling formula

$$\sqrt{2\pi}n^{n+1/2}e^{-n} \leq n! \leq n^{n+1/2}e^{-n+1},$$

we have that

$$\begin{aligned} (2h)^{\frac{N-k}{2}} \binom{3\frac{N-k}{4}}{\frac{N-k}{2}} \frac{N-k}{2} &\leq \frac{yh^{-1}}{2} (2h)^{\frac{yh^{-1}}{2}} \frac{\left(\frac{3yh^{-1}}{4}\right)^{\frac{3yh^{-1}}{4} + \frac{1}{2}} e^{1 - \frac{3yh^{-1}}{4}}}{2\pi \left(\frac{yh^{-1}}{2}\right)^{\frac{yh^{-1}+1}{2}} e^{-\frac{yh^{-1}}{2}} \left(\frac{yh^{-1}}{4}\right)^{\frac{yh^{-1}+2}{4}} e^{-\frac{yh^{-1}}{4}}} \\ &= (3\sqrt{3}h)^{\frac{yh^{-1}}{2}} \sqrt{\frac{3}{2}} \frac{e}{2\pi} (yh^{-1})^{1/2} \xrightarrow{h \rightarrow 0} 0. \end{aligned}$$

Hence, given  $\epsilon > 0$ , it exists  $h_0$  such that  $\forall h, N$  and  $k$  verifying that  $h \leq h_0$ ,  $Nh = \pi$ ,  $kh = x$ ,

$$\sum_{j=\frac{N-k}{2}}^{N-k} a_j^h \leq \epsilon.$$

Now we have to deal with  $I$ , that is, the part  $0 \leq j \leq \frac{N-k}{2}$ . We treat this part of  $s_N$  in a similar way, using again the Stirling formula. We will distinguish the cases  $j$  even and  $j$  odd, although the estimate is deduced exactly in the same way. For  $j$  even we have

$$\begin{aligned} a_j^h &\leq (2h)^j \binom{\frac{N-k+j}{2}}{j} \leq \frac{(2h)^j}{j!} \frac{\left(\frac{N-k+j}{2}\right)^{\frac{N-k+j}{2} + \frac{1}{2}} e^{1 - \frac{N-k+j}{2}}}{\sqrt{2\pi} \left(\frac{N-k-j}{2}\right)^{\frac{N-k-j}{2} + \frac{1}{2}} e^{-\frac{N-k-j}{2}}} \\ &= \frac{e^{1-j}}{\sqrt{2\pi}j!} \left(\frac{y+jh}{y-jh}\right)^{1/2} \left(\frac{y+jh}{y-jh}\right)^{\frac{y-jh}{2h}} (y+jh)^j. \end{aligned}$$

Taking logarithms and using the fact that  $\log(1+x) < x$  we see that  $e^{-j} \left(\frac{y+jh}{y-jh}\right)^{\frac{y-jh}{2h}} < 1$ . Moreover,  $0 \leq jh \leq \frac{y}{2} \Rightarrow y - jh \geq \frac{y}{2}$ , so

$$a_j^h \leq \sqrt{\frac{3}{2\pi}} \left(\frac{3y}{2}\right)^j \frac{e}{j!}, \text{ for } j \text{ even.}$$

Now we consider the case  $j$  odd. Using the same formula,

$$a_j^h \leq (2h)^j \binom{\frac{N-k+j-1}{2}}{j} \leq \frac{e^{1-j}}{\sqrt{2\pi}j!} \left(\frac{y+jh-h}{y-jh-h}\right)^{1/2} \left(1 + \frac{2jh}{y-jh-h}\right)^{\frac{y-jh-h}{2h}} (y+jh-h)^j.$$

Again,  $e^{-j} \left(1 + \frac{2jh}{y-jh-h}\right)^{\frac{y-jh-h}{2h}} < 1$ , while now, the fact that  $\frac{N-k}{2}$  is even and  $j$  is odd tells us that

$$\begin{aligned} j &\leq \frac{N-k}{2} - 1 \Rightarrow jh \leq \frac{y}{2} - h \Rightarrow y - jh - h \geq \frac{y}{2}, \\ j &\leq \frac{N-k}{2} + 1 \Rightarrow jh - h \leq \frac{y}{2}, \end{aligned}$$

therefore

$$a_j^h \leq \sqrt{\frac{3}{2\pi}} \left(\frac{3y}{2}\right)^j \frac{e}{j!}, \text{ for } j \text{ odd.}$$

Hence, by Weierstrass criterion,

$$\lim_{\substack{h \rightarrow 0 \\ Nh=L \\ kh=x}} \sum_{j=0}^{\frac{N-k}{2}} a_j^h = \sum_{j=0}^{\infty} \lim_{\substack{h \rightarrow 0 \\ Nh=L \\ kh=x}} a_j^h = \sum_{j=0}^{\infty} \frac{(1 + \cos(x))^j}{j!} = e^{1+\cos(x)}.$$

Now we have to apply this procedure to the denominator  $t_N$ , but it is easy to check that we can use all the information above setting  $k = 0$  and  $x = 0$ . Thus,

$$b_j^h \xrightarrow[\substack{h \rightarrow 0 \\ Nh=\pi}]{2^j} \int_0^{\pi/2} \int_{x_1}^{\pi/2} \dots \int_{x_{j-1}}^{\pi/2} \sin(2x_1) \sin(2x_2) \dots \sin(2x_j) dx_k \dots dx_2 dx_1,$$

and, again,

$$\begin{aligned} 2^j \int_0^{\pi/2} \int_{x_1}^{\pi/2} \dots \int_{x_{j-1}}^{\pi/2} \sin(2x_1) \sin(2x_2) \dots \sin(2x_j) dx_k \dots dx_2 dx_1 \\ = \left( 2 \int_0^{\pi/2} \sin(2z) dz \right)^j \frac{1}{j!} = \frac{2^j}{j!}, \end{aligned}$$

and this implies that

$$\lim_{\substack{h \rightarrow 0 \\ Nh=L}} t_N = e^2.$$

Finally, we have

$$\lim_{\substack{h \rightarrow 0 \\ Nh=L \\ kh=x}} \omega_k^h = \frac{e^{1+\cos(x)}}{e^2} = e^{\cos(x)-1},$$

exactly the minimizer of the periodic uncertainty principle (8) setting there  $h = 1$  and the initial condition  $\omega(0) = 1$ . Therefore, as we have said above, we recover the minimizing function of the periodic uncertainty principle, since the general case follows the same procedure that this particular case. We have proved then:

**Theorem 5.1.** *Assume  $N \in \mathbb{Z}^+$  and  $h > 0$  are given by the relation  $Nh = L$ . Then, the solution  $\omega = (\omega_k^h)_{k=-N}^N$  of*

$$(23) \quad \begin{cases} (\mathcal{S}_h + \mathcal{A}_h)\omega = 0, \\ \omega_0^h = 1, \end{cases}$$

*converges to the function  $e^{L^2(\cos(\pi x/L)-1)/\pi^2}$  in  $(-L, L)$  in the following sense:  
For  $x \in (-L, L)$ ,*

$$\lim_{\substack{h \rightarrow 0 \\ Nh=L \\ kh=x}} \omega_k^h = e^{L^2(\cos(\pi x/L)-1)/\pi^2}.$$

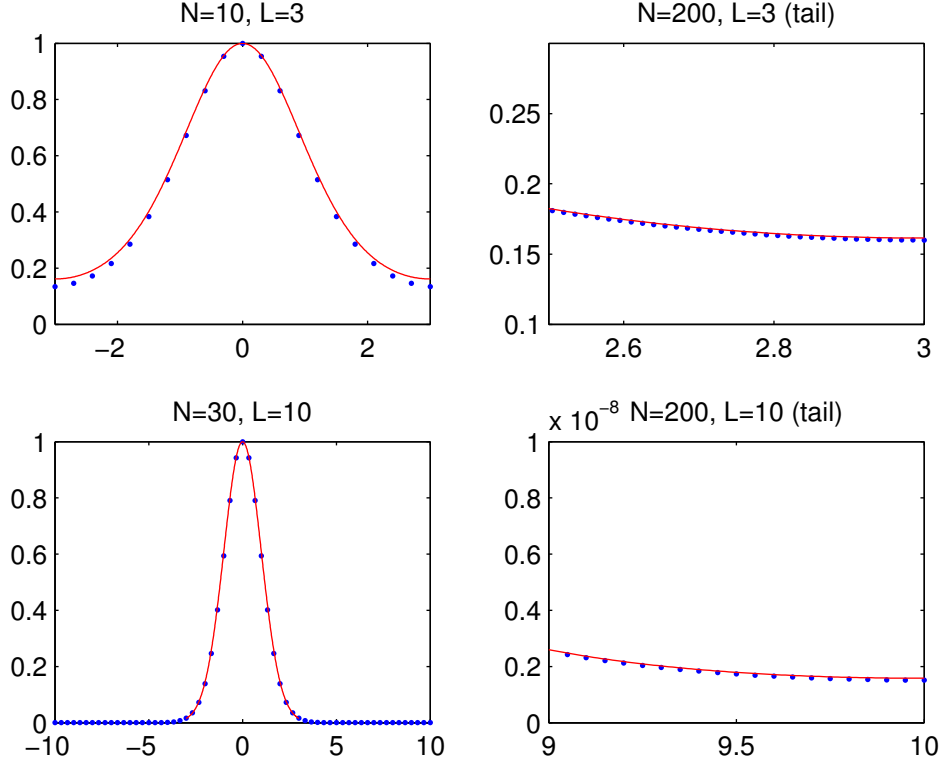


FIGURE 2. Graphic representation of the minimizing sequence and the minimizer of the periodic uncertainty principle in two cases. We see here that when  $L$  is large the minimizing sequence approaches the Gaussian. We also see that the convergence in the tails is slower than in the center of the interval.

5.2. **Periodic case.** In this case we will consider the Hilbert space

$$\mathcal{H}_{per} = \{a = (a_k) : h \sum_{k=-N}^N |a_k|^2 + h \sum_{k=-N}^N |kha_k|^2 < +\infty\},$$

making the identification  $a_{N+1} = a_{-N}$ ,  $a_{-N-1} = a_N$ , and the following symmetric and skew-symmetric operators, represented by the matrices

$$(24) \quad \mathcal{S}_{per} = \begin{bmatrix} -Nh & & 0 \\ & \ddots & \\ 0 & & Nh \end{bmatrix}, \quad \mathcal{A}_{per} = \frac{1}{2h} \begin{bmatrix} 0 & 1 & 0 & \cdots & -1 \\ -1 & 0 & 1 & \cdots & 0 \\ 0 & -1 & 0 & \ddots & 0 \\ & & \ddots & \ddots & \\ 1 & 0 & \cdots & -1 & 0 \end{bmatrix}.$$

Since the operators are represented by a symmetric and a skew-symmetric matrix respectively, the operators are symmetric and skew-symmetric respectively.

The commutator  $[\mathcal{S}_{per}, \mathcal{A}_{per}]$  is represented by the matrix  $\mathcal{S}_{per}\mathcal{A}_{per} - \mathcal{A}_{per}\mathcal{S}_{per}$ , so we have

$$[\mathcal{S}_{per}, \mathcal{A}_{per}]u_k = \begin{cases} Nu_N - \frac{1}{2}u_{-N+1}, & k = -N, \\ -\frac{1}{2}u_{k+1} - \frac{1}{2}u_{k-1}, & k = -N+1, \dots, N-1, \\ Nu_{-N} - \frac{1}{2}u_{N-1}, & k = N, \end{cases}$$

and, after some calculations we have

$$\begin{aligned} \langle -[\mathcal{S}_{per}, \mathcal{A}_{per}]u, u \rangle &= h\Re \sum_{k=-N}^{N-1} u_k \overline{u_{k+1}} - 2Nh\Re(u_N \overline{u_{-N}}) \\ &= h \sum_{k=-N}^N |u_k|^2 - \frac{h^2}{2} \sum_{k=-N}^{N-1} \left| \frac{u_{k+1} - u_k}{h} \right|^2 \\ &\quad - \frac{h}{2} (|u_{-N}|^2 + |u_N|^2 + 4N\Re(u_N \overline{u_{-N}})). \end{aligned}$$

Now we look for the minimizing sequence  $\omega$  that verifies the identity in the last equality. This sequence verifies  $(\mathcal{A}_{per} + \alpha\mathcal{S}_{per})\omega = 0$ , that is

$$(25) \quad \begin{cases} \frac{\omega_{-N+1} - \omega_N}{2h} - \alpha Nh\omega_{-N} = 0, \\ \frac{\omega_{k+1} - \omega_{k-1}}{2h} + \alpha kh\omega_k = 0, & k = -N+1, \dots, N-1, \\ \frac{\omega_{-N} - \omega_{N-1}}{2h} + \alpha Nh\omega_N = 0. \end{cases}$$

We can solve this system and write  $\omega_k$  in terms of  $\omega_0$  using continued fractions. Then, studying the limit of  $\omega_k$  when  $N$  tends to infinity, we can see that

$$\omega_k \xrightarrow{N \rightarrow \infty} \frac{I_k(1/\alpha h^2)}{I_0(1/\alpha h^2)} \omega_0,$$

which was the minimizing sequence of our first uncertainty principle. We do not give the details of this here because it is a bit easier to do that in the next case. Then we have the following result:

**Theorem 5.2.** *For all  $u = (u_k) \in \mathcal{H}_{per}$*

$$\begin{aligned} &|\langle -[\mathcal{S}_{per}, \mathcal{A}_{per}]u, u \rangle| \\ &\leq 2 \left( h \sum_{k=-N}^N |khu_k|^2 \right)^{1/2} \left( h \sum_{k=-N+1}^{N-1} \left| \frac{u_{k+1} - u_{k-1}}{2h} \right|^2 + \left| \frac{u_{-N+1} - u_N}{2h} \right|^2 + \left| \frac{u_{-N} - u_{N-1}}{2h} \right|^2 \right)^{1/2}, \end{aligned}$$

*and the equality is attained for the sequence  $(\omega_k)$  verifying (25). Moreover, when we let  $N$  tend to  $\infty$ , this sequence tends to the minimizer of (6).*

Now we are going to see that we do not have a Virial identity in this finite case. In order to simplify, we set  $h = 1$ .

The equation we consider here is

$$\begin{cases} \partial_t u_k = i(u_{k+1} - 2u_k + u_{k-1}), & k = -N+1, \dots, N-1, \\ \partial_t u_N = i(u_{-N} - 2u_N + u_{N-1}), \\ \partial_t u_{-N} = i(u_{-N+1} - 2u_{-N} + u_N). \end{cases}$$

Differentiating  $\sum_{k=-N}^N |u_k(t)|^2$  we notice that this equation is  $\mathcal{H}_{per}$ -invariant.



Now we differentiate  $F(t) = \sum_{k=-N}^N k^2 |u_k|^2$ , getting

$$\begin{aligned}\dot{F}(t) &= 2\Im \sum_{k=-N+1}^N (1-2k)u_k \overline{u_{k-1}}. \\ \ddot{F}(t) &= 2\Re \left( -2 \sum_{k=-N+1}^{N-1} u_{k+1} \overline{u_{k-1}} + 2 \sum_{-N+2}^N |u_{k-1}|^2 \right. \\ &\quad \left. + (2N-1)(u_{-N} \overline{u_{N-1}} - |u_N|^2 - |u_{-N}|^2 + u_{-N+1} \overline{u_N}) \right).\end{aligned}$$

In the classic and  $\ell_h^2(\mathbb{Z})$  cases,  $\ddot{F}(t) = C \geq 0$ . Furthermore,  $\ddot{F}(t)$  was 8 times the momentum term on the uncertainty principle. This is not the case of the Periodic case, since

$$\ddot{F}(t) = 8 \left( \sum_{k=-N}^N \left| \frac{u_{k+1} - u_{k-1}}{2} \right|^2 + (2N+1) \left( \frac{u_{-N} \overline{u_{N-1}} + u_{-N+1} \overline{u_N} - |u_N|^2 - |u_{-N}|^2}{4} \right) \right).$$

Moreover,  $\ddot{F}(t)$  is not a positive constant. We can take another derivative to check that

$$\ddot{\ddot{F}}(t) = C_N \Im(3u_N \overline{u_{N-1}} - u_{-N} \overline{u_{N2}} + u_{-N+2} \overline{u_N} - 3u_{-N+1} \overline{u_{-N}}) \neq 0,$$

and we also have

$$\begin{aligned}N=3, \quad u(0) &= (0, 1, 0, 0, 0, 1, 0) \Rightarrow \ddot{F}(0) = 8 \geq 0, \\ N=3, \quad u(0) &= (2, 1, 0, 0, 0, 1, 0) \Rightarrow \ddot{F}(0) = -12 \leq 0.\end{aligned}$$

*Remark 5.3.* If we had that  $\ddot{F}(t)$  is the momentum term, then we would have that  $\ddot{\ddot{F}}(t)=0$ .

**5.3. Dirichlet case.** Now we consider the Hilbert space

$$\mathcal{H}_{dir} = \{a = (a_k) : h \sum_{k=-N}^N |a_k|^2 + h \sum_{k=-N}^N |kha_k|^2 < +\infty, \quad a_N = a_{-N} = 0\},$$

and the operators

$$(26) \quad \mathcal{S}_{dir} = \begin{bmatrix} -Nh & & 0 \\ & \ddots & \\ 0 & & Nh \end{bmatrix}, \quad \mathcal{A}_{dir} = \frac{1}{2h} \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ & & \ddots & \ddots & \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

These operators are the same operators we take in (24) but with a slight modification in  $\mathcal{A}_{dir}$  in order to send a sequence in  $\mathcal{H}_{dir}$  to another sequence in  $\mathcal{H}_{dir}$ . Thanks to this, both operators acting on sequences in  $\mathcal{H}_{dir}$  give another sequence in  $\mathcal{H}_{dir}$  and they are respectively symmetric and skew-symmetric.

The uncertainty principle now is very similar to the one we get above, but we have to take into account that the first and the last components of the sequences are zero and then the uncertainty

principle is,  $\forall u \in \mathcal{H}_{dir}$ ,

$$(27) \quad \left| h \sum_{k=-N+1}^{N-1} |u_k|^2 - \frac{h^2}{2} \sum_{k=-N}^{N-1} \left| \frac{u_{k+1} - u_k}{h} \right|^2 \right| = \left| \Re h \sum_{k=-N+1}^{N-1} u_{k+1} \overline{u_k} \right| \\ \leq 2 \left( h \sum_{k=-N+1}^{N-1} |k h u_k|^2 \right)^{1/2} \left( h \sum_{k=-N+1}^{N-1} \left| \frac{u_{k+1} - u_{k-1}}{2h} \right|^2 \right)^{1/2}.$$

Now we want to see who the minimizing sequence is in this inequality. This sequence  $\omega \in \mathcal{H}_{dir}$ , as before, has to verify  $(\alpha \mathcal{S}_{dir} + \mathcal{A}_{dir})\omega = \mathbf{0}$ , that is,  $\omega_N = \omega_{-N} = 0$  and

$$\alpha k h \omega_k + \frac{\omega_{k+1} - \omega_{k-1}}{2h} = 0 \iff \omega_{k+1} + 2\alpha k h^2 \omega_k = \omega_{k-1}, \quad k = -N+1, \dots, N-1.$$

Considering the equation  $k = 0$ , we have that  $\omega_1 = \omega_{-1}$ , and, by induction, we easily see that  $\omega_{-k} = \omega_k$ ,  $k = -N+1, \dots, N-1$ , and, by an iterative process

$$(28) \quad \omega_k = \frac{1}{[2k\alpha h^2, \dots, 2(N-1)\alpha h^2]} \frac{1}{[2(k-1)\alpha h^2, \dots, 2(N-1)\alpha h^2]} \cdots \frac{1}{[2\alpha h^2, \dots, 2(N-1)\alpha h^2]} \omega_0,$$

where

$$[a_0, a_1, \dots, a_n] = a_0 + \frac{1}{a_1 + \frac{1}{\ddots + \frac{1}{a_n}}}.$$

In order to compute the value of each continued fraction, we use again (see Section 5.1) Theorem 149 in [9], and we observe that

$$[2k\alpha h^2, \dots, 2(N-1)\alpha h^2] = \frac{(-1)^{N+k} K_{k-1}(1/\alpha h^2) I_N(1/\alpha h^2) + I_{k-1}(1/\alpha h^2) K_N(1/\alpha h^2)}{(-1)^{N+k+1} K_k(1/\alpha h^2) I_N(1/\alpha h^2) + I_k(1/\alpha h^2) K_N(1/\alpha h^2)}.$$

Since we know that  $I_N(1/\alpha h^2)$  tends to zero and  $K_N(1/\alpha h^2) \simeq CN!$  when  $N$  tends to infinity, we have

$$[2k\alpha h^2, \dots, 2(N-1)\alpha h^2] \xrightarrow{N \rightarrow \infty} \frac{I_{k-1}(1/\alpha h^2)}{I_k(1/\alpha h^2)},$$

hence, from (28),

$$\omega_k \xrightarrow{N \rightarrow \infty} \frac{I_k(1/\alpha h^2)}{I_{k-1}(1/\alpha h^2)} \frac{I_{k-1}(1/\alpha h^2)}{I_{k-2}(1/\alpha h^2)} \cdots \frac{I_1(1/\alpha h^2)}{I_0(1/\alpha h^2)} \omega_0 = \frac{I_k(1/\alpha h^2)}{I_0(1/\alpha h^2)} \omega_0.$$

Therefore we recover the minimizing sequence of the first uncertainty principle we have seen here.

**Theorem 5.3.** *For all  $u = (u_k) \in \mathcal{H}_{dir}$  the inequality (27) holds, and the equality is attained for the sequence  $(\omega_k)$  given by (28). Moreover, when we let  $N$  tend to  $\infty$ , this sequence tends to the minimizer of (6).*

Wondering about the existence of an analogue of (2) in this Dirichlet case (we simplify again  $h = 1$ ), we consider a solution to the discrete Schrödinger equation

$$\begin{cases} \partial_t u_k = i(u_{k+1} - 2u_k + u_{k-1}), & k = -N+2, \dots, N-2, \\ \partial_t u_{N-1} = i(-2u_{N-1} + u_{N-2}), \\ \partial_t u_{-N+1} = i(u_{-N+2} - 2u_{-N+1}). \end{cases}$$

It is easy to check that this equation is  $\mathcal{H}_{dir}$ -invariant. Moreover,

$$\dot{F}(t) = 2\Im \sum_{k=-N+2}^{N-1} (2k-1)u_k \overline{u_{k-1}}.$$

Taking another derivative,

$$\begin{aligned} \ddot{F}(t) = 2\Re \left( -2 \sum_{k=-N+2}^{N-2} u_{k+1} \overline{u_{k-1}} + 2 \sum_{k=-N+3}^{N-1} |u_{k-1}|^2 - (2N-3)|u_{N-1}|^2 \right. \\ \left. - (2N-3)|u_{-N+1}|^2 \right). \end{aligned}$$

In the classic and  $\ell_h^2(\mathbb{Z})$  cases,  $\ddot{F}(t) = C \geq 0$ . Furthermore,  $\ddot{F}(t)$  was 8 times the momentum term on the uncertainty principle. This is not the case of the Dirichlet case, since

$$\ddot{F}(t) = 8 \left( \sum_{k=-N+1}^{N-1} \left| \frac{u_{k+1} - u_{k-1}}{2} \right|^2 - (2N-2) \left| \frac{u_{N-1}}{2} \right|^2 - (2N-2) \left| \frac{u_{-N+1}}{2} \right|^2 \right).$$

Moreover,  $\ddot{F}(t)$  is not a positive constant. We can take another derivative to check that

$$\ddot{\ddot{F}}(t) = C_N \Im(u_{N-2} \overline{u_{N-1}} + u_{-N+2} \overline{u_{-N+1}}) \neq 0,$$

where  $C_N$  is a constant which depends on  $N$ . We also have

$$\begin{aligned} N=3, \quad u(0) = (0, 1, 0, 0, 0, 1, 0) &\Rightarrow \ddot{F}(0) = -12 \leq 0, \\ N=3, \quad u(0) = (0, 1, 2, 0, 0, 1, 0) &\Rightarrow \ddot{F}(0) = 4 \geq 0. \end{aligned}$$

*Remark 5.4.* Even if we had that  $\ddot{F}(t)$  is the momentum term, then  $\ddot{\ddot{F}}(t)$  would not be zero, as we can see differentiating the momentum term, being this a difference between the Dirichlet case and the Periodic case.

*Remark 5.5.* The non-existence of a convex parabola like (3) in these finite cases makes sense, since, as we have said in the introduction, in the continuous case, when the periodic Schrödinger equation is considered, there is no equivalent to Theorem 1.1.

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